Purposeful Academic Classes for Excelling Students Program
(Department of Education, Western Australia)

Mathematics Methods Units 3 & 4

Session 3

Definite integrals

3.2.10 examine the area problem and use sums of the form $\sum_i f(x_i)\ \delta x_i$ to estimate the area under the curve $y = f(x)$
3.2.11 identify the definite integral $\int_a^b f(x)dx$ as a limit of sums of the form $\sum_i f(x_i)\ \delta x_i$
3.2.12 interpret the definite integral $\int_a^b f(x)dx$ as area under the curve $y = f(x)$ if $f(x) > 0$
3.2.13 interpret $\int_a^b f(x)dx$ as a sum of signed areas
3.2.14 apply the additivity and linearity of definite integrals

Applications of integration

3.2.18 calculate total change by integrating instantaneous or marginal rate of change
3.2.19 calculate the area under a curve
3.2.20 calculate the area between curves
3.2.21 determine displacement given velocity in linear motion problems
3.2.22 determine positions given linear acceleration and initial values of position and velocity.

General discrete random variables

3.3.1 develop the concepts of a discrete random variable and its associated probability function, and their use in modelling data
3.3.2 use relative frequencies obtained from data to obtain point estimates of probabilities associated with a discrete random variable
3.3.3 identify uniform discrete random variables and use them to model random phenomena with equally likely outcomes
3.3.4 examine simple examples of non-uniform discrete random variables
3.3.5 identify the mean or expected value of a discrete random variable as a measurement of centre, and evaluate it in simple cases
3.3.6 identify the variance and standard deviation of a discrete random variable as measures of spread, and evaluate them using technology
3.3.7 examine the effects of linear changes of scale and origin on the mean and the standard deviation
3.3.8 use discrete random variables and associated probabilities to solve practical problems

Bernoulli distributions

3.3.9 use a Bernoulli random variable as a model for two-outcome situations
3.3.10 identify contexts suitable for modelling by Bernoulli random variables
3.3.11 determine the mean p and variance $p(1-p)$ of the Bernoulli distribution with parameter p
3.3.12 use Bernoulli random variables and associated probabilities to model data and solve practical problems

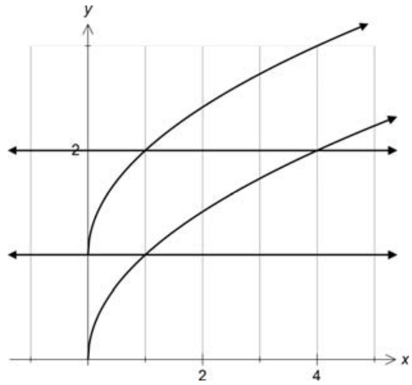
Binomial distributions

3.3.13 examine the concept of Bernoulli trials and the concept of a binomial random variable as the number of ‘successes’ in n independent Bernoulli trials, with the same probability of success p in each trial
3.3.14 identify contexts suitable for modelling by binomial random variables
3.3.15 determine and use the probabilities $P(X = x) = \binom{n}{x} p^x(1-p)^{n-x}$ associated with the binomial distribution with parameters n and p; note the mean np and variance $np(1-p)$ of a binomial distribution
3.3.16 use binomial distributions and associated probabilities to solve practical problems

Area under a Curve

Worked Example 1 **Calculator Free**

The region R is defined as the region trapped between the curves $y = \sqrt{x}$, $y = 1 + \sqrt{x}$ and the lines y = 1 and y = 2. Calculate the area of region R.



$$A = \left(\int_0^1 1 + \sqrt{x} - 1 \, dx \right) + \left((2 \times 3) - \int_1^4 \sqrt{x} \, dx \right)$$

$$= \left(\left[\frac{2x^{\frac{3}{2}}}{3} \right]_0^1 \right) + \left(6 - \left[\frac{2x^{\frac{3}{2}}}{3} \right]_1^4 \right)$$

$$= \frac{2}{3} + 6 - \left[\frac{16}{3} - \frac{2}{3} \right] = 2$$

$y = \sqrt{x} \Rightarrow x = y^2$. Also, $y = 1 + \sqrt{x} \Rightarrow x = (y - 1)^2$.

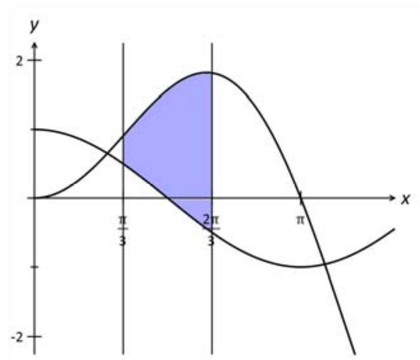
$$A = \int_1^2 y^2 - (y-1)^2 \, dy = \left[\frac{y^3}{3} - \frac{(y-1)^3}{3} \right]_1^2 = \left[\frac{8}{3} - \frac{1}{3} \right] - \left[\frac{1}{3} - 0 \right] = 2$$

Worked Example 2 **Calculator Free**

(a) Determine $\frac{d}{dx}(x \cos x)$.

$$\frac{d}{dx}(x \cos x) = \cos x - x \sin x$$

(b) The shaded region in the accompanying diagram is trapped between the curves $y = x \sin x$, $y = \cos x$ and the lines $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$. Calculate the area of the shaded region.



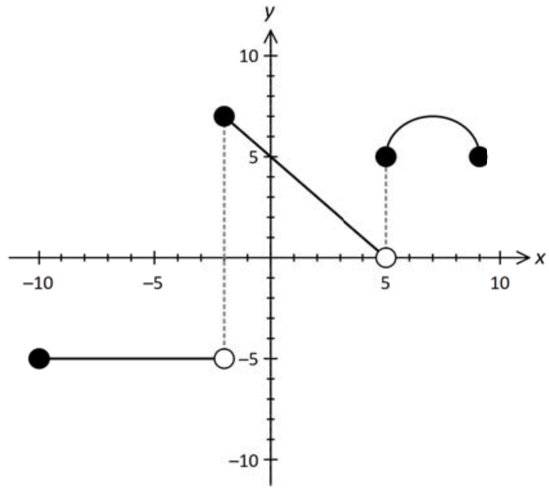
$$\text{Area} = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} x \sin x - \cos x \, dx$$

$$= -\left[x \cos x \right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$$

$$= -\left[\left(\frac{-\pi}{3} \right) - \left(\frac{\pi}{6} \right) \right] = \frac{\pi}{2}$$

Worked Example 3 **Calculator Assumed**

The diagram below shows the graph of $y = f(x)$ for $-10 \le x \le 9$. The graph consists of the horizontal line $y = -5$ for $-10 \le x < 5$, the line $y = -x + 5$ for $-2 \le x < 5$ and the semicircle with centre at (7, 5) and radius 2 for $5 \le x \le 9$.



(a) Determine the value of $\int_{-10}^{5} f(x)\,dx$.

$$\int_{-10}^{5} f(x)\,dx = \int_{-10}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$$
$$= 8 \times (-5) + \frac{1}{2} \times 7 \times 7$$
$$= -40 + 24.5 = -15.5$$

(b) Determine the area of the region trapped between the curve, the x-axis and the lines $x = -10$ and $x = 5$.

$$\text{Area} = \left| \int_{-10}^{-2} f(x)\,dx \right| + \int_{-2}^{5} f(x)\,dx$$
$$= 40 + 24.5 = 64.5$$

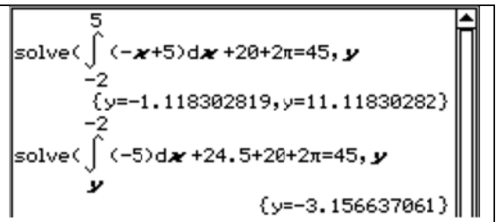
(c) Determine the value of k if $\int_{k}^{9} f(x)\,dx = 45$ where the constant $-10 \le k \le 9$.

For $-2 \le k \le 9$

$$\int_{k}^{5} -x + 5\,dx + [20 + \frac{1}{2} \times \pi \times 2^2] = 45 \quad k = -1.11$$

For $-2 \le k \le -5$

$$\int_{k}^{5} -5\,dx + 24.5 + 20 + \frac{1}{2} \times \pi \times 2^2 = 45 \quad k = -3.16$$



Area Functions

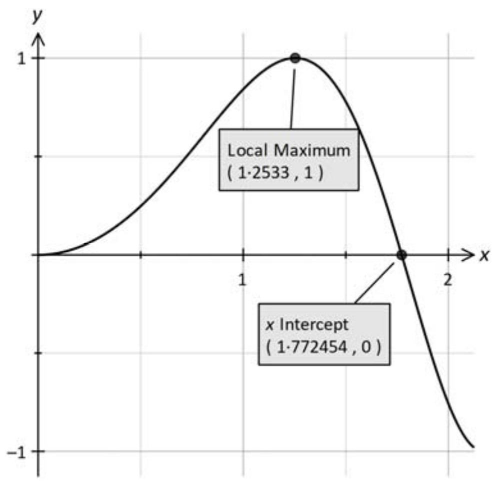
Worked Example 4 **Calculator Assumed**

A curve has equation given by $y = f(x) = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 2$.

(a) Use your calculator to complete the table below.

x	0	0.5	1	1.5	2
$f(x)$	0	0.04148	0.3103	0.7782	0.8048

(b) In the axes provided, sketch the graph of $y = \sin(x^2)$ for $0 \le x \le 2$.



(c) Use your graph in (b) to help you determine the x-coordinate of the turning point on the graph of $y = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 2$. Hence, determine the coordinates of the turning point on the graph of $y = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 2$.

For TP: $\dfrac{dy}{dx} = \sin(x^2) = 0$

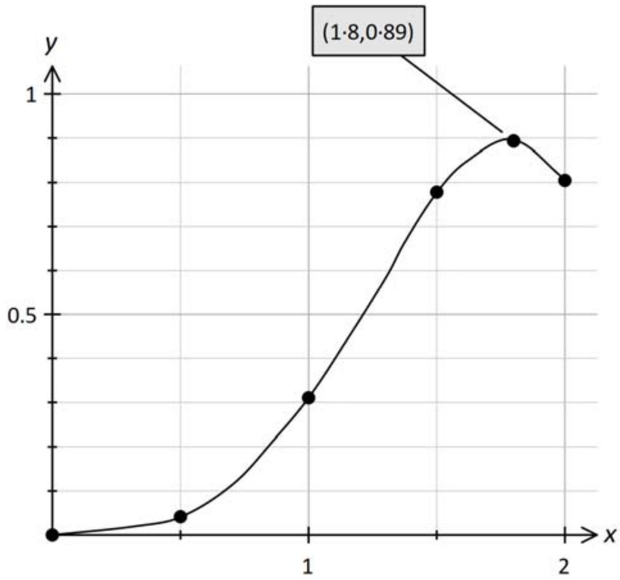
From graph: $\dfrac{dy}{dx} = 0$ for $x = 1.77$

$\Rightarrow \quad y = \int_0^{1.77} \sin(t^2)\,dt \approx 0.89.$

Hence, TP is (1.77, 0.89)

(d) Hence, or otherwise, on the axes provided below, sketch the graph of

$y = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 2$. Indicate clearly the stationary point on the curve.



Net Change

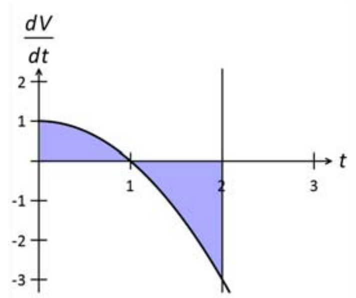
Worked Example 5 Calculator Assumed

The rate at which water evaporates, in litres per day, from a swimming pool is given by $\frac{dV}{dt} = \sqrt{1+e^{-t}}$ where V is the volume in litres and t is the time in days. Write down a definite integral that, when evaluated, would give the volume of water lost from evaporation between the 3rd and 6th day. Use your calculator to evaluate this integral.

$$V = \int_2^6 \sqrt{1+e^{-t}}\ dt$$
$$= 4.0653 \text{ litres}$$

Worked Example 6 Calculator Free

The instantaneous rate with which the amount of liquid, V litres, in a holding tank, changes with respect to time t minutes, is modelled by $\frac{dV}{dt} = 1 - t^2$. The sketch of $\frac{dV}{dt}$ against t is shown in the accompanying diagram.



(a) Explain what happens at $t = 1$ minute.

The rate of liquid flow is instantaneously zero.
Liquid stops flowing in and begins to flow out.
Volume of liquid in the tank achieves a local maximum.

(b) Find the amount of liquid in the tank after 2 minutes, if initially there were 5 litres in the tank.

$$\text{Net Change in tank after 2 minutes} = \int_0^2 1 - t^2\ dt$$
$$= \left[t - \frac{t^3}{3}\right]_0^2$$
$$= \left[2 - \frac{8}{3}\right] = -\frac{2}{3}$$

Hence, amount left in tank = $5 - \frac{2}{3} = \frac{13}{3}$ litres

Rectilinear Motion

- displacement $\underset{\text{integrate}}{\overset{\text{differentiate}}{\rightleftarrows}}$ velocity $\underset{\text{integrate}}{\overset{\text{differentiate}}{\rightleftarrows}}$ acceleration

- The net change in displacement in the interval $a \le t \le b$ is $\int_a^b v\,dt$.

- The total distance travelled in the interval $a \le t \le b$ is the area of the region trapped between the curve $v = f(t)$, the lines $t = a$, $t = b$ and the t-axis.
  Where use of a CAS/graphic calculator is permitted,

  Total distance travelled = $\int_a^b |v|\,dt$.

Worked Example 7 **Calculator Free**

A particle P moves along a straight line. Its velocity, $\frac{dx}{dt}$ ms^{-1}, t seconds after passing a fixed point O, is given by $\frac{dx}{dt} = kt + C$, where k and C are constants. The change in displacement in the first 4 seconds is 8 m. P reverses direction at $t = 3$ seconds. Find the values of k and C.

P reverses direction at $t = 3$ $\Rightarrow$ $\left.\frac{dx}{dt}\right|_{t=3} = 0$

$$3k + C = 0$$

$$C = -3k$$

Hence, $\frac{dx}{dt} = kt + -3k = k(t - 3)$

Change in displacement in the first 4 seconds is 8 m

$$\Rightarrow \quad k\int_0^4 t - 3\,dt = 8$$

$$k\left[\frac{(t-3)^2}{2}\right]_0^4 = 8$$

$$k\left[\frac{1}{2} - \frac{9}{2}\right] = 8$$

$$k = -2$$

$$C = 6$$

Worked Example 8 **Calculator Assumed**

A particle P travels in a straight line. The point O is a fixed point on this line. P starts with an initial displacement of $\ln\sqrt{2}$ from O with an initial velocity of -1 cm/h. Its acceleration after t hours ($0 \le t < \frac{3\pi}{4}$), is given by $a = \dfrac{1}{\cos^2\left(t-\frac{\pi}{4}\right)}$ cm/h^2.

(a) Use the result $\int \frac{1}{\cos^2 x}\,dx = \tan x + C$ to determine the velocity of P at time $t = \frac{\pi}{4}$ hours.

$$v = \int \frac{1}{\cos^2\left(t-\frac{\pi}{4}\right)}\,dt = \tan\left(t-\frac{\pi}{4}\right) + C$$

$$v(0) = -1 \quad \Rightarrow \quad C = 0$$

$$\Rightarrow \quad v = \tan\left(t-\frac{\pi}{4}\right) \quad \Rightarrow \quad v\left(\frac{\pi}{4}\right) = 0$$

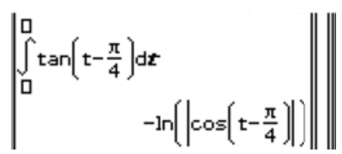
(b) Determine the displacement of P at time $t = \frac{\pi}{4}$ hours.

$$x = \int \tan\left(t-\frac{\pi}{4}\right)dt = \int \frac{\sin\left(t-\frac{\pi}{4}\right)}{\cos\left(t-\frac{\pi}{4}\right)}\,dt$$

$$= -\ln\left[\cos\left(t-\frac{\pi}{4}\right)\right] + D$$

$$x(0) = \ln\sqrt{2} \quad \Rightarrow \quad D = 0$$

$$x\left(\frac{\pi}{4}\right) = 0$$



(c) Describe the motion of P in the time just before and just after $t = \frac{\pi}{4}$ hours.

Just before $t = \frac{\pi}{4}$, P is located right of O and travels towards O.

Just after $t = \frac{\pi}{4}$, P is located right of O and travels away from O.

Changes direction at O at $t = \frac{\pi}{4}$ and starts to move away from O.

(d) Describe what happens to the displacement and velocity of P as $t \to \frac{3\pi}{4}$ hours.

P is infinitely far from O and
velocity of P increases infinitely!

Worked Example 9 **Calculator Assumed**

A particle is moving along a straight line. Its velocity v (in ms^{-1}) at x (metres), is given by $v^2 = 24 - 8x - 2x^2$.

(a) Find all values of x for which the particle is at rest.

$$2x^2 + 8x - 24 = 0$$
$$x^2 + 4x - 12 = 0$$
$$(x + 6)(x - 2) = 0$$
$$x = -6,\ 2 \text{ metres}$$

(b) Determine $\frac{dv}{dx}$.

$$v = \pm\sqrt{24 - 8x - 2x^2}$$
$$\frac{dv}{dx} = \pm\left(\frac{-8-4x}{2\sqrt{24-8x-2x^2}}\right)$$

(c) Use the chain rule to determine $\frac{dv}{dt}$, the acceleration of the particle in terms of x.

Chain Rule: $\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$

Hence, $\frac{dv}{dt} = \pm\left(\frac{-8-4x}{2\sqrt{24-8x-2x^2}}\right) \times \frac{dx}{dt}$

$$= \pm\left(\frac{-8-4x}{2\sqrt{24-8x-2x^2}}\right) \times \pm\sqrt{24-8x-2x^2}$$
$$= \pm(4 + 2x) \text{ ms}^{-2}.$$

(d) Find the maximum speed of the particle.

At maximum speed, $\frac{dv}{dt} = 0 \Rightarrow x = -2$.

Hence, max. speed $= \sqrt{24+16-8}$

$$= \sqrt{32} = 4\sqrt{2} \text{ ms}^{-1}.$$

Worked Example 10 **Calculator Assumed**

Penny is driving her car at 21 ms^{-1} when she suddenly sees an object on the road directly ahead. The instant she applies the brakes her car is 40 metres from the object. The application of the brakes creates a constant deceleration of $5ms^{-2}$ until she hits the object. Let displacement, $x(t)$, represent the distance travelled by the car from the moment the brakes are applied. If the car does not skid and continues travelling in a straight line toward the object, use the acceleration equation, $a(t) = -5$, and calculus methods to calculate the speed (in km per hour) at which the car hits the object.

$a = \frac{d^2x}{dt^2} = -5 \quad \Rightarrow \quad \frac{dx}{dt} = -5t + C$

$t = 0, \frac{dx}{dt} = 21 \quad \Rightarrow \quad \frac{dx}{dt} = -5t + 21$

$x = -\frac{5t^2}{2} + 21t + K$

$t = 0, x = 0 \Rightarrow K = 0 \quad x = -\frac{5t^2}{2} + 21t$

At $x = 40$, $-\frac{5t^2}{2} + 21t = 40 \quad \Rightarrow \quad t = 2.9194$

Hence, $\frac{dx}{dt} = 6.403$ ms^{-1}.

Therefore, speed = 23.1 kmh^{-1}

Discrete Random Variables

- A **Discrete Random Variable** (DRV) is a function that assigns each outcome of a statistical experiment a number.
- The **probability distribution function** for X, $p(x)$, is a function that *assigns each number that X takes a probability value*. This value is obtained by considering the probability of occurrence of the equivalent event. The probability distribution for a DRV, X details the probability of X assuming each value within its range.
- $p(x)$ is a **probability distribution function** for a discrete random variable X if:
  - $p(x) \geq 0$ for all x in the range of X and $\sum p(x) = 1$.
- For a discrete random variable X with probability distribution function $p(x)$:
  - The mean or expected value of X, $\mu = E(X) = \sum \left[x \times p(x) \right]$.
  - The variance of X, $\sigma^2 = Var(X) = \sum \left[(x-\mu)^2 \times p(x) \right] = E(X^2) - [E(X)]^2$.
    where $E(X^2) = \sum \left[x^2 \times p(x) \right]$
  - The standard deviation of X = $\sqrt{Var(X)}$.

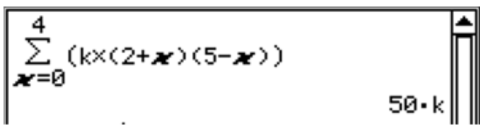
Worked Example 11 **Calculator Assumed**

Let X: No. of cars owned per family in a certain suburb. The probability distribution of X is given by $P(X = x) = k\,(2 + x)(5 - x)$ for $x = 0, 1, 2, 3, 4$.

(a) Find the value of k.

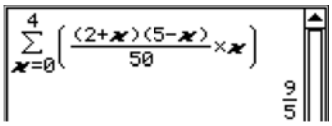
$$\sum_{x=0}^{4} k(2+x)(5-x) = 50k = 1$$

$$k = \frac{1}{50}$$



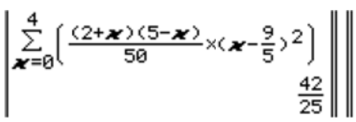
(b) Find μ, the expected number of cars owned per family.

$$E(X) = \sum_{x=0}^{4} \left[\frac{1}{50}(2+x)(5-x) \times x \right] = \frac{9}{5}$$



(c) Calculate σ, the standard deviation associated with the mean for X.

$$Var(X) = \sum_{x=0}^{4} \left[\frac{1}{50}(2+x)(5-x) \times \left(x - \frac{9}{5} \right)^2 \right] = \frac{42}{25}$$



$$STD(X) = \frac{\sqrt{42}}{5}$$

Worked Example 12 Calculator Assumed

A committee of five students is to be selected from a group of six female students and seven male students. Define X: Number of female students in this committee.

(a) Find the probability distribution for X.

X = 0, 1, 2, 3, 4, 5

$P(X = 0) = P(0 \text{ female \& 5 males}) = \dfrac{\binom{6}{0}\binom{7}{5}}{\binom{13}{5}} = \dfrac{7}{429}$

$P(X = 1) = P(1 \text{ female \& 4 males}) = \dfrac{\binom{6}{1}\binom{7}{4}}{\binom{13}{5}} = \dfrac{70}{429}$

$P(X = 2) = P(2 \text{ females \& 3 males}) = \dfrac{\binom{6}{2}\binom{7}{3}}{\binom{13}{5}} = \dfrac{175}{429}$

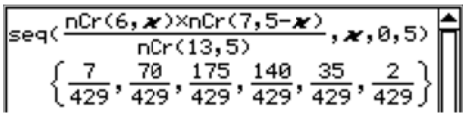
$P(X = 3) = P(3 \text{ females \& 2 males}) = \dfrac{\binom{6}{3}\binom{7}{2}}{\binom{13}{5}} = \dfrac{140}{429}$

$P(X = 4) = P(4 \text{ females \& 1 male}) = \dfrac{\binom{6}{4}\binom{7}{1}}{\binom{13}{5}} = \dfrac{35}{429}$

$P(X = 5) = P(5 \text{ females \& 0 male}) = \dfrac{\binom{6}{5}\binom{7}{0}}{\binom{13}{5}} = \dfrac{2}{429}$

OR

$P(X = x) = \dfrac{\binom{6}{x}\binom{7}{5-x}}{\binom{13}{5}}$ for x = 0, 1, 2, 3, 4, 5.

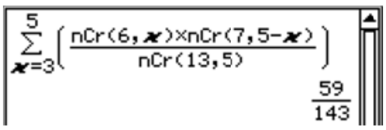


(b) Find the probability that there are at least as many females than males in the committee.

$P(X \geq 3) = P(X = 3) + P(X = 4) + P(X = 5)$

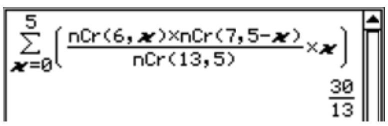
$= \dfrac{140}{429} + \dfrac{35}{429} + \dfrac{2}{429}$

$= \dfrac{177}{429} = \dfrac{59}{143}$



(c) Find the expected number of females in the committee.

$E(X) = \dfrac{30}{13} \approx 2.3$



Worked Example 13 Calculator Assumed

Box P has three balls numbered 1, 2, and 3 respectively while box Q has six balls numbered 1, 2, 3, 4, 5 and 6 respectively.

- Allan draws two balls from box P with replacement and records A, the sum of the numbers on the balls drawn.
- Ben draws one ball from box Q and notes B, the number on the ball drawn.

If A = B, then Allan pays Ben \$2. If A is less than B, the Allan pays Ben \$5.
If A exceeds the B, the Ben pays Allan \$5.

(a) Calculate the probability that Allan has to pay Ben \$2.

A	1	2	3
1	2	3	4
2	3	4	5
3	4	5	6

A	2	3	4	5	6
P(A = a)	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

B	1	2	3	4	5	6
P(B = b)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Let T: Amount of money received by Allan

$$P(T = -2) = P(A = B) = \frac{1}{6}\times\left(\frac{1}{9}+\frac{2}{9}+\frac{3}{9}+\frac{2}{9}+\frac{1}{9}\right) = \frac{1}{6}$$

(b) Calculate the amount of money Allan expects to receive.

Let T: Amount of money received by Allan

T	Equivalent event	P(T = t)
–2	A = B	$\frac{1}{6}$
–5	A < B	$\left(\frac{1}{9}\times\frac{4}{6}\right)+\left(\frac{2}{9}\times\frac{3}{6}\right)+\left(\frac{3}{9}\times\frac{2}{6}\right)+\left(\frac{2}{9}\times\frac{1}{6}\right)=\frac{18}{54}$
5	A > B	$\left(\frac{1}{9}\times\frac{1}{6}\right)+\left(\frac{2}{9}\times\frac{2}{6}\right)+\left(\frac{3}{9}\times\frac{3}{6}\right)+\left(\frac{2}{9}\times\frac{4}{6}\right)+\left(\frac{1}{9}\times\frac{5}{6}\right)=\frac{27}{54}$

$E(T) = (-2)\times\frac{1}{6} + (-5)\times\frac{18}{54} + 5\times\frac{27}{54} = 0.5$ $\Rightarrow$ Allan expects to receive \$0.50.

Worked Example 14 **Calculator Assumed**

The table below shows the probability of a person dying between age x and age $x + 1$ years. For example, the probability of a 25 year old dying before he reaches the age of 26 is 0.0008.

Age x years	Probability of dying during the year
25	0.00080
26	0.00082
27	0.00085
28	0.00088
29	0.00092

An insurance company sells life insurance coverage in multiples of $1 000 at an annual cost of $25 per thousand dollars of coverage for persons aged between 25 and 29 inclusive.
For example, a 25 year old person who buys a life insurance coverage of $100 000 will pay an annual premium of 100 × $25 = $2 500. If the insured person dies within the year, his/her estate will receive the amount insured.

The insurance company has 200 clients aged 25 with coverage of $100 000 each, 400 clients aged 28 each with coverage $150 000 and 300 clients aged 29 each with coverage $200 000.
Determine the expected returns for the company, showing clearly how you arrived at your answer.

For 25 year old ($100 000):

Return X	–97 500	2 500
Probability	0.000 8	0.999 2

E(X) = –97 500 × 0.0008 + 2 500 × 0.992 = $2 420

For 28 year old ($150 000):

Return X	– 146 250	3 750
Probability	0.000 88	0.999 12

E(X) = – 146 250 × 0.000 88 + 3 750 × 0.999 12 = $3 618

For stock 29 year old ($200 000):

Return X	– 195 000	5 000
Probability	0.000 92	0.999 08

E(X) = – 195 000 × 0.000 92 + 5 000 × 0.999 08 = $4 816

Expected return for Company = 200 × 2 420 + 400 × 3 618 + 300 × 4 816
= $3 376 000

Bernoulli Variables

- A Bernoulli Trial is a statistical experiment where the outcomes may be grouped into a "success" category and a "failure" category.
- A Bernoulli variable assigns the "success" category the numerical value of 1 and the "failure" category the numerical value of 0.
- If X is a Bernoulli variable with parameter *p*, (*p* is the probability of success) then:
  - the probability distribution function of X is $p(x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \end{cases}$
  - the mean for X, $\mu = E(X) = p$
    the variance for X, $\sigma^2 = p(1-p) = pq$ here q = P(Failure) = $1 - p$.

The Binomial Distribution

Let X: Number of successes in *n* independent Bernoulli Trials
where P(success) = *p* *[constant for each trial]*

- then, X is called a Binomial Random Variable with parameters *n* and *p*.
- The probability distribution for X is given by:

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r} \qquad \text{where } r = 0, 1, 2, 3 \ldots, n$$

- The mean for X, E(X) = np and the variance for X, Var(X) = npq.

Worked Example 15 Calculator Assumed

(a) 19% of workers in work-place bring their own lunch from home to work every work day. A worker is randomly chosen from this work-place. Define X = 1 if the worker brings lunch from home every work day and X = 0 otherwise. Find the expected value of X and its associated standard deviation.

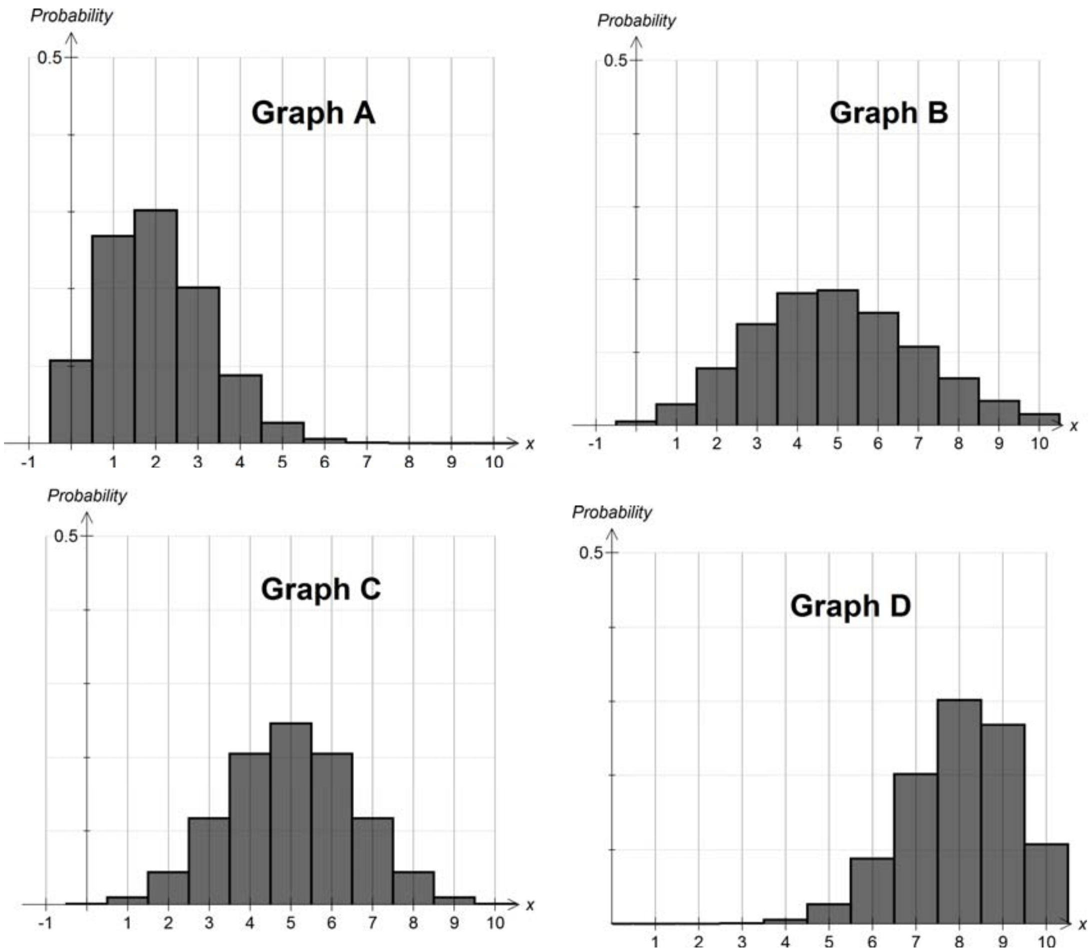
> P(X = 1) = 0.19 and P(X = 0) = 0.81
> E(X) = 0.19
> VAR(X) = $0.19 \times 0.81 = 0.1539 \Rightarrow$ STD(X) = 0.3923

(b) A box contains 15 red balls and 5 green balls. A ball is drawn from the box, its colour noted and it is ***not*** returned to the box. This process is repeated 10 times. Define X: No. of red balls drawn in the 10 draws. Determine with reasons if X is a binomial variable.

> P(red ball in first trial) = 0.75 for each trial.
> P(red ball in second trial) = $\frac{15}{19}$ or $\frac{14}{19} \neq 0.75$
> Hence, trials are not independent Bernoulli Trials.
> Therefore, X is not a binomial variable.

Worked Example 16 Calculator Free

The diagram below shows the graphs of the probability distributions of several distributions.



Match the graphs with the probability distribution from the list below:

$X_1 \sim$ Binomial (n = 10, p = 0.3) $X_2 \sim$ Binomial (n = 10, p = 0.5)
$X_3 \sim$ Binomial (n = 20, p = 0.5) $X_4 \sim$ Binomial (n = 50, p = 0.1)
$X_5 \sim$ Binomial (n = 10, p = 0.8) $X_6 \sim$ Binomial (n = 20, p = 0.1)

Give clear reasons for your answer.

Graph A: Most likely value = 2 $\Rightarrow$ mean is close to 2.
Hence, Graph A is matched with $X_6 \sim$ Binomial (n = 20, p = 0.1).

Graph B & Graph C: Most likely value = 5 $\Rightarrow$ mean is close to 5.
Hence, possibilities are:
$X_2 \sim$ Binomial (n = 10, p = 0.5) or $X_4 \sim$ Binomial (n = 50, p = 0.1)
P(X = 0) is higher in Graph B than Graph C.
Hence, Graph B is matched with $X_4 \sim$ Binomial (n = 50, p = 0.1)
and Graph C is matched with $X_2 \sim$ Binomial (n = 10, p = 0.5).

Graph D: Most likely value = 8 $\Rightarrow$ mean is close to 8.
Hence, graph D is matched to $X_5 \sim$ Binomial (n = 10, p = 0.8).

Worked Example 17 **Calculator Assumed**

Only 5% of all applicants are successful in being admitted to a special army operations unit.

(a) Find the probability that in a batch of 50 applicants, no more than the expected number of successful applicants are successful.

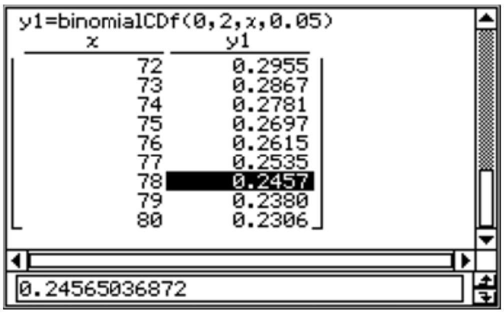
> X: No. of successful applicants out of 50.
> P(applicant is successful) = 0.05 (Fixed)
> Hence, $X \sim B(n = 50, p = 0.05)$
> $E(X) = 50 \times 0.05 = 2.5$
> $P(X \le 2) \approx 0.5405$

(b) In 20 batches of 50 applicants each, find the probability that at least 10 of these batches each have no more successful applicants than the expected number within the batch.

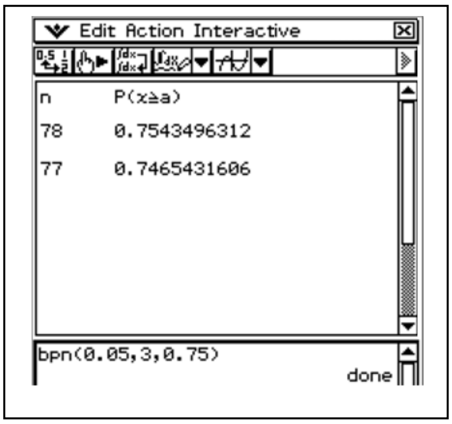
> Y: No. of batches out of 20 with no more successful applicants than the expected number within the batch.
> P(success) = 0.540533
> Hence, $Y \sim B(n = 20, p = 0.540533)$
> $P(Y \ge 10) = 0.7225$

(c) At least 3 applicants need to be chosen from a group of *n* applicants. Find the minimum value of *n* if the probability of achieving this is to be at least 75%.

> W: No. of successful applicants out of *n*.
> P(applicant is successful) = 0.05 (Fixed)
> Hence, $W \sim B(n, p = 0.05)$.
>
> $P(X \ge 3) \ge 0.75 \Rightarrow P(X \le 2) \le 0.25$
>
> $$0.95^n + (n \times 0.05 \times 0.95^{n-1}) + \left(\frac{n \times (n-1)}{2} \times 0.05^2 \times 0.95^{n-2}\right) \le 0.25$$
>
> $n \ge 78$
>
> OR



OR



Worked Example 18 **Calculator Assumed**

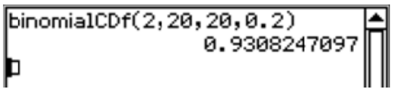
20% of clients of a hair dresser fail to turn up for their scheduled appointments. On a certain day, 20 appointments were made to see the hair dresser.

(a) Find the probability that all 20 clients turn up for their respective appointments.

X: No. of clients that fail to turn up out of 20.
P(client fails to turn up) = 0.2 (Fixed)
Hence, X ~ B(n = 20, p = 0.2)
P(X = 0) = 0.8^{20} = 0.01153

(b) Find the probability that at least 2 clients fail to turn up for their appointments.

P(X ≥ 2) = 0.9308



(c) Find the probability that on two separate days with 20 appointments each, at least 4 clients fail to turn up for their appointments.

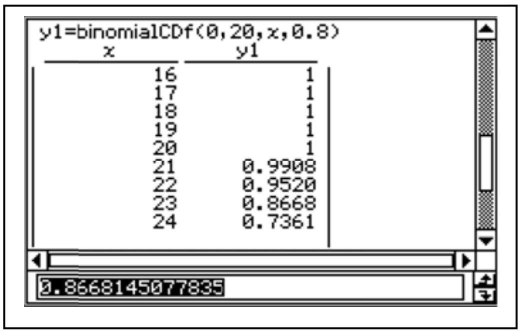
Y: No. of clients that fail to turn up out of 40.
P(client fails to turn up) = 0.2 (Fixed)
Hence, Y ~ B(n = 40, p = 0.2)
P(Y ≥ 4) = 0.9715

(d) On any given day, the hair dresser has sufficient time to see 20 clients. To improve efficiency, the hair dresser decides to make more than 20 appointments a day. However, there must be a 90% chance that the hairdresser will have sufficient time to attend to all the clients that turn up. Find the greatest number of appointments that the hairdresser can make.

W: No. of clients that turn up out of n.
P(client turns up) = 0.8 (Fixed)
Hence, W ~ B(n, p = 0.8)
P(W ≤ 20) = 0.90

Trial & Error
n = 21: P(W ≤ 20) = 0.9908
n = 22: P(W ≤ 20) = 0.9520
n = 23: P(W ≤ 20) = 0.8668

Hence, n = 22.



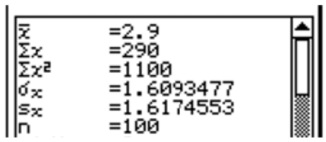
Worked Example 19 **Calculator Assumed**

One hundred samples of ten drivers (between the ages of 17 and 25 years) each were surveyed as to whether they had ever been booked for speeding. The accompanying table shows the distribution of persons per sample who had been booked for speeding for the hundred samples taken.

No. of persons booked for speeding	0	1	2	3	4	5	6	7	8	9	10
No. of samples	4	15	22	28	19	7	2	1	1	1	0
Expected number of samples	3	13	24	27	19	9	3	1	0	0	0

(a) Calculate the mean number of persons booked for speeding per sample.

Mean = 2.9



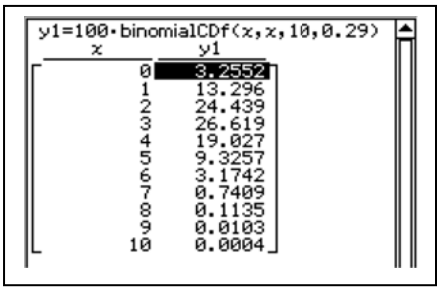
Define X: Number of persons per sample who had been booked for speeding.
If X is a binomial variable, then X ~ B(10, *p*) where *p* is the probability that a driver has been booked for speeding.

(b) Use your answer in (a) to calculate the value of *p*.

Mean for X = $10p = 2.9$
$p = 0.29$

(c) Use your answer in (b) to complete the table given above. Hence, determine with reasons if the data given may be modelled by a Binomial variable.

There is a reasonably good match between the actual number of samples and the expected number of samples for a given number of people booked per sample.
Hence, a Binomial variable could be used.



(d) An alternative method to calculate *p* would be to use the relative frequency for zero drivers booked per sample. Use this method to calculate *p*.

X ~ B(10, *p*).

From table, $P(X = 0) = \dfrac{4}{100} = 0.04. \quad \Rightarrow (1-p)^{10} = 0.04$

$\Rightarrow \quad p = 0.2752$

Worked Example 20 Calculator Assumed

1% of USB drives manufactured by a factory are defective. Several USB drives are selected without replacement.

(a) Write mathematical expressions for the probabilities of each of the following events.

(i) 10 USB drives need to be picked before picking the first defective drive. (i.e. the first defective USB drive is the 10th USB drive picked)

Required Probability = $0.99^9 \times 0.01 \approx 0.009\,135$

(ii) There are exactly 2 defective USB drives in the first 10 USB drives picked.

Required Probability = $\binom{10}{2} 0.99^8 \times 0.01^2 \approx 0.004\,152$

(iii) The third defective USB drive is the 10th USB drive picked.

Required Probability = $\binom{9}{2} 0.99^7 \times 0.01^2 \times 0.01 \approx 0.000\,033\,554$

(iv) Two defective drives are among 5 USB drives picked from a selection that contained 3 defective and 17 non-defective USB drives.

Required Probability = $\dfrac{\binom{3}{2}\binom{17}{3}}{\binom{20}{5}} = \dfrac{5}{38}$

(b) Calculate the mean number of drives that need to be picked before picking the first defective USB drive. State the accompanying standard deviation.

Define X: No. of drives required before picking 1st defective drive.
X = 1, 2, 3, 4, ….. .

P(X = n) = $0.99^{n-1} \times 0.01$

E(X) = $\sum_{n=1}^{\infty} \left(0.99^{n-1} \times 0.01 \times n\right)$ = 100

Var(X) = $\sum_{n=1}^{\infty} \left(0.99^{n-1} \times 0.01 \times (n-100)^2\right)$ = 9 900

Std(X) = $30\sqrt{11} \approx 99.5$